2013 Leaving Cert Higher Level
Official Sample Paper 1

Section A **Concepts and Skills** **150 marks**

Question 1 **(25 marks)**

(a) $w = -1 + \sqrt{3}i$ is a complex number, where $i^2 = -1$.

(i) Write w in polar form.

We have $|w| = \sqrt{(-1)^2 + \sqrt{3}^2} = \sqrt{4} = 2$. Also, if $\arg(w) = \theta$, then $\tan(\theta) = \frac{\sqrt{3}}{-1} = -\sqrt{3}$ and θ lies in the second quadrant (from the diagram). Therefore $\theta = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$radians. So

$$w = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

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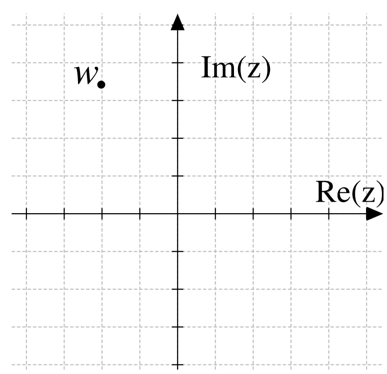


(ii) Use De Moivre's Theorem to solve the equation $z^2 = -1 + \sqrt{3}i$. Give your answer(s) in rectangular form.

Suppose that the polar form of z is given by $z = r(\cos\theta + i\sin\theta)$. Then

$$[r(\cos\theta + i\sin\theta)]^2 = 2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$$

By De Moivre's Theorem this is equivalent to

$$r^2(\cos(2\theta) + i\sin(2\theta)) = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right).$$

Therefore $r^2 = 2$ and $2\theta = \frac{2\pi}{3} + 2n\pi$, $n \in \mathbb{Z}$. So $r = \sqrt{2}$ and $\theta = \frac{\pi}{3} + n\pi$. We get two distinct solutions (corresponding to $n = 0$ and $n = 1$).

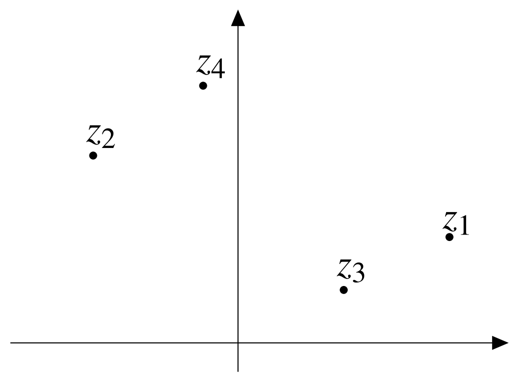
$$z_1 = \sqrt{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \sqrt{2}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{1}{\sqrt{2}} + i\sqrt{\frac{3}{2}}$$

and

$$z_2 = \sqrt{2}\left(\cos(\frac{\pi}{3} + \pi) + i\sin(\frac{\pi}{3} + \pi)\right) = \sqrt{2}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -\frac{1}{\sqrt{2}} - i\sqrt{\frac{3}{2}}$$

(b) Four complex numbers z_1, z_2, z_3 and z_4 are shown on the Argand diagram. They satisfy the following conditions:

$$\begin{aligned} z_2 &= iz_1 \\ z_3 &= kz_1, \text{ where } k \in \mathbb{R} \\ z_4 &= z_2 + z_3 \end{aligned}$$



The same scale is used on both axes.

(i) Identify which number is which by labelling the points on the diagram.

(ii) Write down the approximate value of k.

Answer: $\boxed{\frac{1}{2}}$

Explanation: Multiplication by i rotates a complex number by 90° anticlockwise about the origin - so z_2 is obtained by rotating z_1 through 90° about the origin.

Since $z_3 = kz_1$, we must have 0, z_1 and z_3 being collinear.

Since $z_4 = z_2 + z_3$, we must have 0, z_2, z_4 and z_3 forming a parallelogram.

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Question 2 **(25 marks)**

(a) Prove by induction that $\displaystyle\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ for any $n \in \mathbb{N}$.

First we check that the statement is true for $n = 1$. The sum of the first 1 natural numbers is 1, and when $n = 1$ we have $\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$. So the statement is true for $n = 1$.

Now suppose that the statement is true for some $n \geq 1$. Remember that $\displaystyle\sum_{r=1}^{n} r = 1 + 2 + \cdots + n$. So

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Now, add $n+1$ to both sides and we get

$$\begin{aligned}
1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\
&= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\
&= \frac{n(n+1) + 2(n+1)}{2} \\
&= \frac{(n+2)(n+1)}{2} \\
&= \frac{(n+1)(n+2)}{2}
\end{aligned}$$

So the sum of the first $n+1$ natural numbers is $\frac{(n+1)((n+1)+1)}{2}$, which completes the induction step. Therefore, by induction, the statement is true for all natural numbers n.

(b) State the range of values for which the series $\displaystyle\sum_{r=2}^{\infty} (4x-1)^r$ is convergent, and write the infinite sum in terms of x.

This is a geometric series i.e. a series of the form $T_n = ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots$ where $a = 1$ and $r = 4x - 1$.

However, given that it starts from r=2, this series is missing the first two terms a and ar (1 and $4x-1$). If this series is convergent we must have $|4x-1| < 1$ which means

$$\begin{aligned} -1 < 4x - 1 < 1 \\ 0 < 4x < 2 \\ 0 < x < \frac{1}{2} \end{aligned}$$

This is the required range.

The sum to infinity of a geometric series is given by $S_\infty = \dfrac{a}{1-r}$ where $|r| < 1$. Since this series is missing the first two terms we get

$$\begin{aligned} S_\infty &= \frac{1}{1-(4x-1)} - 1 - (4x-1) \\ &= \frac{1}{2-4x} - 4x \\ &= \frac{1}{2-4x} - \frac{(2-4x)4x}{2-4x} \\ &= \frac{1-8x+16x^2}{2-4x} \end{aligned}$$

Question 3 **(25 marks)**

A cubic function f is defined for $x \in \mathbb{R}$ as

$$f : x \mapsto x^3 + (1-k^2)x + k, \text{ where } k \text{ is a constant.}$$

(a) Show that $-k$ is a root of f.

Substituting $-k$ for x we obtain

$$\begin{aligned} f(-k) &= (-k)^3 + (1-k^2)(-k) + k \\ &= -k^3 - k + k^3 + k \\ &= 0 \end{aligned}$$

Therefore $-k$ is a root of f.

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(b) Find, in terms of k, the other two roots of f.

Since $-k$ is a root of f we know, by the Factor Theorem, that $(x+k)$ is a factor of $f(x)$. Now we carry out long division to find the other factor.

$$
\begin{array}{rrrrrrrrr}
 & x^2 & - & kx & + & 1 & & \\
\cline{2-8}
x+k \,\big) & x^3 & + & 0x^2 & + & (1-k^2)x & + & k \\
 & x^3 & + & kx^2 & & & & \\
\cline{2-4}
 & & - & kx^2 & + & (1-k^2)x & & \\
 & & - & kx^2 & - & k^2x & & \\
\cline{3-6}
 & & & & & x & + & k \\
 & & & & & x & + & k \\
\cline{6-8}
 & & & & & & & 0
\end{array}
$$

So
$$x^3+(1-k^2)x+k=(x+k)(x^2-kx+1).$$
Therefore the other two roots of f are solutions of the equation
$$x^2-kx+1=0.$$
Using the quadratic formula we get
$$x=\frac{k\pm\sqrt{k^2-4}}{2}.$$
So the other two roots of f are
$$\frac{k+\sqrt{k^2-4}}{2} \text{ and } \frac{k-\sqrt{k^2-4}}{2}.$$

(c) Find the set of values of k for which f has exactly one real root.

From the solution to part (b), we see that f has exactly one real root if and only if $k^2-4<0$. This is equivalent to $k^2<4$ or
$$-2<k<2$$

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Question 4 **(25 marks)**

(a) Solve the simultaneous equations:

$$\begin{aligned} 2x+8y-3z &= -1 \\ 2x-3y+2z &= 2 \\ 2x+y+z &= 5. \end{aligned}$$

We can subtract the second equation from the first:

$$\begin{array}{rcl} 2x+8y-3z &=& -1 \\ 2x-3y+2z &=& 2 \\ \hline 11y-5z &=& -3 \end{array}$$

Similarly, we subtract the third equation from the first:

$$\begin{array}{rcl} 2x+8y-3z &=& -1 \\ 2x+y+z &=& 5 \\ \hline 7y-4z &=& -6 \end{array}$$

Now we solve the simultaneous equations

$$\begin{array}{rcl} 11y-5z &=& -3 \\ 7y-4z &=& -6 \end{array}$$

Multiply the first by 7, the second by 11 and subtract:

$$\begin{array}{rcl} 77y-35z &=& -21 \\ 77y-44z &=& -66 \\ \hline 9z &=& 45 \end{array}$$

Therefore $z = \frac{45}{9} = 5$. Now substitute $z = 5$ into $7y-4z = -6$ to get $7y-4(5) = -6$ or $7y = -6+20 = 14$ Therefore $y = 2$.
Finally substitute $y = 2$ and $z = 5$ into $2x+8y-3z = -1$ to get $2x+8(2)-3(5) = -1$ or $2x = -1-8(2)+3(5) = -2$ So $x = -1$.
So the solution is

$$x = -1, y = 2, z = 5.$$

Now we can check this by substituting into the original equations and verifying that they are all true:

$$\begin{array}{rcl} 2(-1)+8(2)-3(5) &=& -1 \\ 2(-1)-3(2)+2(5) &=& 2 \\ 2(-1)+(2)+(5) &=& 5. \end{array}$$

(b) The graphs of the functions $f : x \mapsto |x-3|$ and $g : x \mapsto 2$ are shown in the diagram.

(i) Find the co-ordinates of the points A, B, C and D.

D is on the y-axis, so its x-co-ordinate is 0. Now $f(0) = |0-3| = |-3| = 3$. So $D = (0,3)$.
$C = (,0)$ (on the x-axis), so we solve $|x-3| = 0$ to find the x-co-ordinate. Now $|x-3| = 0 \Leftrightarrow x-3 = 0 \Leftrightarrow x = 3$. So $C = (3,0)$.
A and B both have y-co-ordinate 2, so we solve $|x-3| = 2$. Now $|x-3| = 2 \Leftrightarrow \pm(x-3) = 2$. So either

$$(x-3) = 2 \text{ or } -(x-3) = 2.$$

In the first case $x = 5$ and in the second case $-x+3 = 2$ or $x = 1$. So $A = (1,2)$ and $B = (5,2)$.

$$\begin{array}{ll} A = (1,2) & B = (5,2) \\ C = (3,0) & D = (0,3) \end{array}$$

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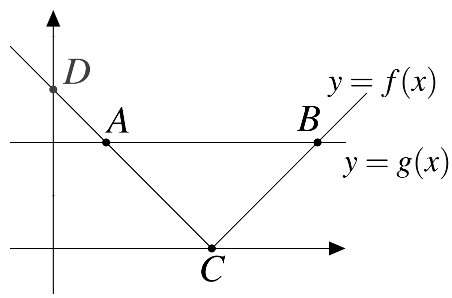

(ii) Hence, or otherwise, solve the inequality $|x-3| < 2$.

The solution set of the inequality corresponds to the values of x for which the graph of f is below the graph of g. From the diagram and calculations above, we see that the solution set is

$$1 < x < 5.$$

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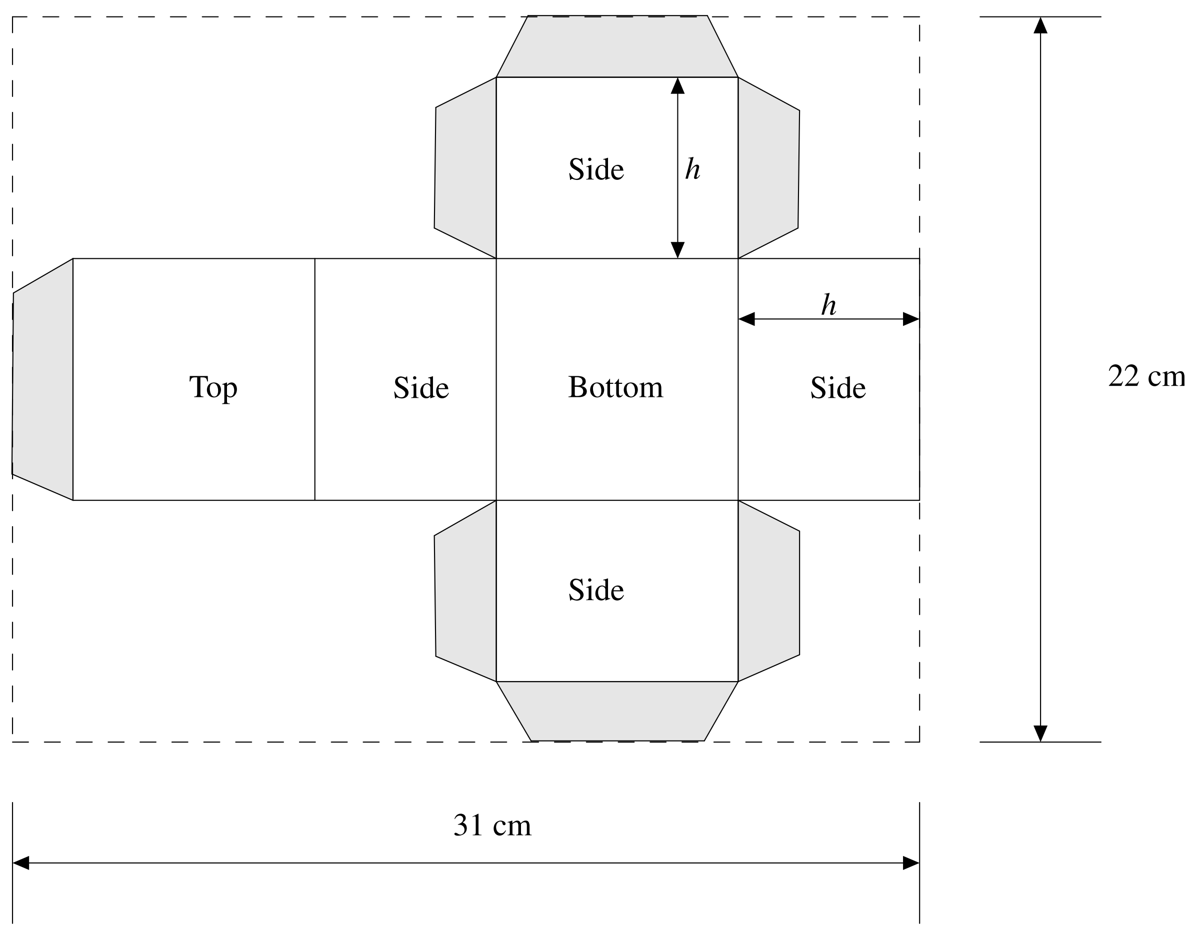
Question 5 **(50 marks)**

A company has to design a rectangular box for a new range of jellybeans. The box is to be assembled from a single piece of cardboard, cut from a rectangular sheet measuring 31 cm by 22 cm. The box is to have a capacity (volume) of 500 cm^3 .

The net for the box is shown below. The company is going to use the full length and width of the rectangular piece of cardboard. The shaded areas are flaps of width 1 cm which are needed for assembly. The height of the box is h cm, as shown on the diagram.



(a) Write the dimensions of the box, in centimetres, in terms of h.

Let l be the length of the box and let w be the width of the box, both in centimetres. Then by adding up dimensions as we move left to right across the diagram above, we see that $1+l+h+l+h=31$. Therefore, by isolating l in this equation we obtain

$$l = 15 - h.$$

Going top to bottom, we see that $1+h+w+h+1=22$ and by isolating w, we see that

$$w = 20 - 2h.$$

Therefore

height =	h cm
length =	$15-h$ cm
width =	$20-2h$ cm

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(b) Write an expression for the capacity of the box in cubic centimetres, in terms of h.

Capacity = length $\times$ width $\times$ height $= (15-h)(20-2h)h = 2h^3 - 50h^2 + 300h\text{cm}^3$.

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(c) Show that the value of h that gives a box with a square bottom will give the correct capacity.

The bottom of the box is square if and only if length = width. In other words, if and only if $15-h=20-2h$. This is equivalent to $h=5$. From the solution to part (b), we calculate that, when $h=5$, the capacity of the box will be $(15-5)(20-2(5))5 = 10(10)(5) = 500\text{cm}^3$, as required.

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(d) Find, correct to one decimal place, the other value of h that gives a box of the correct capacity.

We must solve $2h^3 - 50h^2 + 300h = 500$, or

$$2h^3 - 50h^2 + 300h - 500 = 0.$$

From part (c), we know that $h = 5$ is one solution. Therefore, by the Factor Theorem, $(h-5)$ is a factor of $2h^3 - 50h^2 + 300h - 500$. Factorising yields

$$2h^3 - 50h^2 + 300h - 500 = (h-5)(2h^2 - 40h + 100).$$

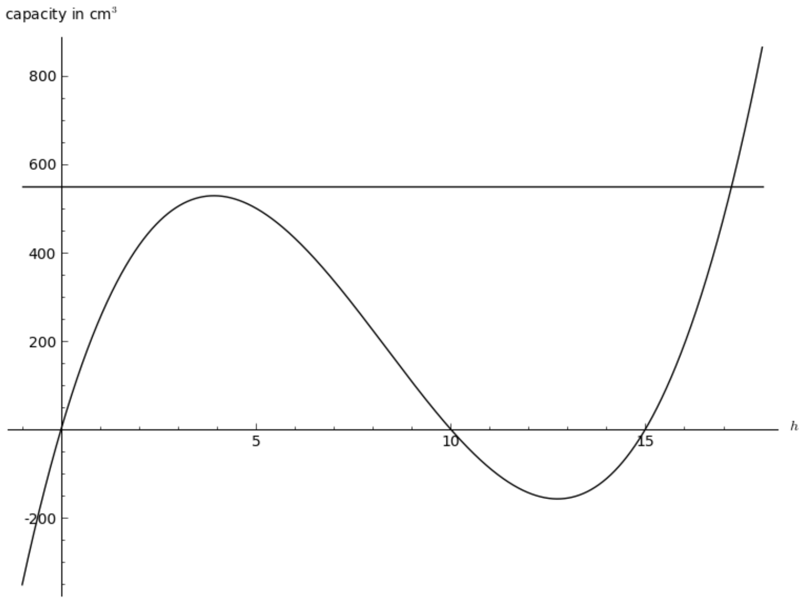
Now, we solve $2h^2 - 40h + 100 = 0$ using the quadratic formula. So

$$h = \frac{40 \pm \sqrt{40^2 - 4(2)(100)}}{2(2)} = \frac{40 \pm \sqrt{800}}{4} = 10 \pm \sqrt{50}$$

So, correct to one decimal place, $h = 17.1$ or $h = 2.9$.
Now, however, we observe that since the length of the box is $15 - h$, we must have $15 - h > 0$ or $h < 15$. Therefore $h \neq 17.1$. So the other value of h that gives the correct capacity is 2.9cm.

(e) The client is planning a special "10% extra free" promotion and needs to increase the capacity of the box by 10%. The company is checking whether they can make this new box from a piece of cardboard the same size as the original one (31cm × 22cm). They draw the graph below to represent the box's capacity as a function of h. Use the graph to explain why it is not possible to make the larger box from such a piece of cardboard.

Explanation:



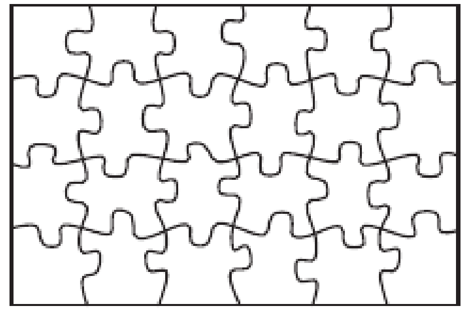
The capacity of the new box will be $1.1 \times 500 = 550\text{cm}^3$. On the diagram above we have drawn a horizontal line representing the equation

$$\text{Capacity} = 550.$$

We can see from the diagram that this horizontal line only meets the cubic curve at one point and that the h-co-ordinate of that point is greater than 15.
However, as we observed above, for any box constructed as described in the question, we must have $h < 15$. Therefore it is not possible to make the bigger box from the same piece of cardboard as before.

Question 6 **(25 marks)**

A rectangular jigsaw puzzle has pieces arranged in rows. Each row has the same number of pieces. For example, the picture on the right shows a 4×6 jigsaw puzzle - there are four rows with 6 pieces in each row.



Every piece of the puzzle is either an *edge piece* or an *interior piece*. The puzzle shown has 16 edge pieces and 8 interior pieces.

Investigate the number of edge pieces and the number of interior pieces in an $m \times n$ jigsaw puzzle, for different values of m and n. Start by exploring some particular cases, and then attempt to answer the questions that follow, with justification.

Initial Exploration:

2×2: In this case there are 4 jigsaw pieces and all of those are edge pieces.
3×2: In this case there are 6 jigsaw pieces and all of those are edge pieces too. In fact any jigsaw which is $m \times 2$ or $2 \times n$ consists entirely of edge pieces and no interior ones.
3×3: There are 9 jigsaw pieces and 8 of those are edge pieces, 1 is an interior piece.
4×3: There are 12 jigsaw pieces and 10 of those are edge pieces, 2 are interior pieces.
4×4: There are 16 jigsaw pieces and 12 of those are edge pieces, 4 are interior pieces.

(a) How do the number of edge pieces and the number of interior pieces compare in cases where either $m \leq 4$ or $n \leq 4$?

For an $m \times n$ jigsaw the edge pieces can be counted by looking at the perimeter (be careful not to count the corner pieces twice).

The left side of the jigsaw has m pieces, as does the right side.

The top side has n pieces but we've already counted the 2 corner pieces which means there are $n-2$ pieces left on the top to count (the same is true on the bottom of the jigsaw). So

$$\begin{aligned} \text{number of edge pieces} &= m+m+(n-2)+(n-2) \\ &= 2m+2n-4 \end{aligned}$$

To calculate the interior pieces we can take the number of edge pieces, $2m+2n-4$ away from the total, mn to get

$$\begin{aligned} \text{number of interior pieces} &= mn-(2m+2n-4) \\ &= mn-2m-2n+4 \end{aligned}$$

Now we can factorise to get

$$\text{Number of interior pieces} = (m-2)(n-2)$$

[Aside: looking at the diagram, this makes sense as it represents:
(the number of edge pieces on the left side, less 2) multiplied by (the number of edge pieces on the top side, less 2).]

If $m \leq 4$, then

$$(m-2)(n-2) \leq (4-2)(n-2) = 2n-4$$

So the number of interior pieces is at most

$$2n-4$$

Comparing this with the formula for the number of edge pieces, we see that, given m must be positive, then:

$$2n-4 < 2m+2n-4$$

So the number of interior pieces is less than the number of edge pieces.

Similarly, if $n \leq 4$ the number of interior pieces is less than the number of edge pieces.

(b) Show that if the number of edge pieces is equal to the number of interior pieces, then

$$m = 4 + \frac{8}{n-4}$$

From part (a) we know that the number of edge pieces is equal to the number of interior pieces if and only if

$$2m + 2n - 4 = mn - 2m - 2n + 4.$$

Now rearranging this equation gives

$$mn - 4m - 4n + 8 = 0.$$

We add 8 to both sides to get

$$mn - 4m - 4n + 16 = 8$$

and now we can factorise the left hand side to get

$$(m-4)(n-4) = 8$$

Now divide across by $n-4$ ...

$$m - 4 = \frac{8}{n-4}$$

... and bring the 4 to the right hand side to get

$$m = 4 + \frac{8}{n-4}$$

as required.

(c) Find all cases in which the number of edge pieces is equal to the number of interior pieces.

We need to solve the equation from part (b)

$$m = 4 + \frac{8}{n-4}$$

We know that n and m must be whole numbers since there can only be a whole number of pieces along each side. So the fraction $\dfrac{8}{n-4}$ must be a whole number i.e. $n-4$ must divide evenly into 8.

This means that $n-4$ must be one of $8, 4, 2, 1, -1, -2, -4, -8$ since these are the only whole numbers that divide evenly into 8. We can solve each of these cases separately:

$$\begin{aligned}
n-4 &= 8 & n &= 12,\ m = 5 && (1)\\
n-4 &= 4 & n &= 8,\ m = 6 && (2)\\
n-4 &= 2 & n &= 6,\ m = 8 && (3)\\
n-4 &= 1 & n &= 5,\ m = 12 && (4)\\
n-4 &= -1 & n &= 3,\ {\color{red} m = -4} && (5)\\
n-4 &= -2 & n &= 2,\ {\color{red} m = 0} && (6)\\
n-4 &= -4 & {\color{red} n} &{\color{red} = 0},\ m = 2 && (7)\\
n-4 &= -8 & {\color{red} n} &{\color{red} = -4},\ m = 3 && (8)
\end{aligned}$$

These last four solutions can be ignored since m and n must be positive (there must be a positive whole number of pieces along the side).

This leaves four cases in which the number of edge pieces is equal to the number of interior pieces. They are:

$$\begin{aligned}
n &= 12 & m &= 5 && (1)\\
n &= 8 & m &= 6 && (2)\\
n &= 6 & m &= 8 && (3)\\
n &= 5 & m &= 12 && (4)
\end{aligned}$$

(d) Determine the circumstances in which there are *fewer* interior pieces than edge pieces. Describe fully all such cases.

Using the information from part (a) we get the following inequality

$$\text{number of interior pieces} < \text{number of edge pieces}$$
$$mn - 2m - 2n + 4 < 2m + 2n - 4$$

Now rearrange to get

$$mn - 4m - 4n + 8 < 0.$$

Add 8 to both sides to get

$$mn - 4m - 4n + 16 < 8$$

and factorise the left hand side to get

$$(m-4)(n-4) < 8 \tag{5}$$

Now we know that m and n are positive whole numbers.
Also, we already know from part (a) that if either $m \leq 4$ or $n \leq 4$, then there are fewer interior pieces than edge pieces.
Thus we only have to consider the case where $m \geq 5$ and $n \geq 5$.

If $m = 5$ then the inequality (5) above becomes $n - 4 < 8$, or $n < 12$.
If $m = 6$, it becomes $2(n-4) < 8$ or $n - 4 < 4$ or $n < 8$.
If $n = 7$, then we get $n < 20/3$.

We continue in this way and we get the following cases:

m	n
5	5,6,7,8,9,10,11
6	5,6,7
7	5,6
8	5
9	5
10	5
11	5

In summary there are fewer interior pieces than edge pieces when:
m or n is less than or equal to 4,
or
when (m,n) is one of the following: $(5,5)$, $(5,6)$, $(5,7)$, $(5,8)$, $(5,9)$, $(5,10)$, $(5,11)$, $(6,5)$, $(6,6)$, $(6,7)$, $(7,5)$, $(7,6)$, $(8,5)$, $(9,5)$, $(10,5)$, $(11,5)$.

Section B	Contexts and Applications	150 marks

Question 7 **(50 marks)**

Old syllabus

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Question 8 **(50 marks)**

(a) Find $\int (\sin 2x + e^{4x} dx.)$

$$\begin{aligned} \int (\sin 2x + e^{4x} dx.) &= \frac{-\cos 2x}{2} + \frac{e^{4x}}{4} + c \\ &= -\frac{1}{2}\cos 2x + \frac{1}{4}e^{4x} + c \end{aligned}$$

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(b) Let $y = \dfrac{\cos x + \sin x}{\cos x - \sin x}$.

(i) Find $\dfrac{dy}{dx}$.

Use the quotient rule: let $u = \cos x + \sin x$ and $v = \cos x - \sin x$. Then

$$\frac{du}{dx} = -\sin x + \cos x \qquad \text{and} \qquad \frac{dv}{dx} = -\sin x - \cos x.$$

So

$$\begin{aligned}
\frac{dy}{dx} &= \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \\
&= \frac{(\cos x - \sin x)(-\sin x + \cos x) - (\cos x + \sin x)(-\sin x - \cos x)}{(\cos x - \sin x)^2} \\
&= \frac{-\cos x \sin x + \cos^2 x + \sin^2 x - \cos x \sin x - (-\cos x \sin x - \cos^2 x - \sin^2 x - \cos x \sin x)}{(\cos x - \sin x)^2} \\
&= \frac{-\cos x \sin x + 1 - \cos x \sin x + \cos x \sin x + \cos^2 x + \sin^2 x + \cos x \sin x}{(\cos x - \sin x)^2} \\
&= \frac{-\cos x \sin x + 1 - \cos x \sin x + \cos x \sin x + 1 + \cos x \sin x}{(\cos x - \sin x)^2} \\
&= \frac{2}{(\cos x - \sin x)^2}
\end{aligned}$$

(ii) Show that $\dfrac{dy}{dx} = 1 + y^2$

$$
\begin{aligned}
1 + y^2 &= 1 + \left(\frac{\cos x + \sin x}{\cos x - \sin x}\right)^2 \\
&= 1 + \frac{(\cos x + \sin x)^2}{(\cos x - \sin x)^2} \\
&= \frac{(\cos x - \sin x)^2}{(\cos x - \sin x)^2} + \frac{(\cos x + \sin x)^2}{(\cos x - \sin x)^2} \\
&= \frac{(\cos x - \sin x)^2 + (\cos x + \sin x)^2}{(\cos x - \sin x)^2} \\
&= \frac{(\cos^2 x + \sin^2 x - 2\cos x \sin x) + (\cos^2 x + \sin^2 x + 2\cos x \sin x)}{(\cos x - \sin x)^2} \\
&= \frac{1 - 2\cos x \sin x + 1 + 2\cos x \sin x)}{(\cos x - \sin x)^2} \\
&= \frac{2}{(\cos x - \sin x)^2} \\
&= \frac{dy}{dx}
\end{aligned}
$$

(c) **(i)** Find in terms of a and b,

$$I = \int_a^b \frac{\cos x}{1+\sin x} dx$$

Use substitution: Let $u = 1+\sin x$. Then

$$\begin{aligned} \frac{du}{dx} &= \cos x \\ \frac{du}{\cos x} &= dx \end{aligned}$$

Now the integral I becomes

$$\begin{aligned} I &= \int_a^b \frac{\cos x}{u} \frac{du}{\cos x} \\ &= \int_a^b \frac{1}{u} du \\ &= \ln u|_a^b \\ &= \ln(1+\sin x)|_a^b \\ &= \ln(1+\sin b) - \ln(1+\sin a) \end{aligned}$$

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(ii) Find in terms of a and b,

$$J = \int_a^b \frac{\sin x}{1+\cos x} dx$$

Use substitution: Let $u = 1+\cos x$. Then

$$\begin{aligned} \frac{du}{dx} &= -\sin x \\ -\frac{du}{\sin x} &= dx \end{aligned}$$

Now the integral J becomes

$$\begin{aligned} J &= \int_a^b \frac{\sin x}{u}\left(-\frac{du}{\sin x}\right) \\ &= -\int_a^b \frac{1}{u} du \\ &= -\ln u|_a^b \\ &= -\ln(1+\cos x)|_a^b \\ &= -\ln(1+\cos b)+\ln(1+\cos a) \end{aligned}$$

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(iii) Show that if $a+b=\frac{\pi}{2}$ then $I=J$.

If $a+b=\frac{\pi}{2}$ then $b=\frac{\pi}{2}-a$ and $a=\frac{\pi}{2}-b$.

So

$$\begin{aligned} I &= \ln(1+\sin b)-\ln(1+\sin a) \\ &= \ln\left(1+\sin\left(\frac{\pi}{2}-a\right)\right)-\ln\left(1+\sin\left(\frac{\pi}{2}-b\right)\right) \\ &= \ln(1+\cos(-a))-\ln(1+\cos(-b)) \\ &= \ln(1+\cos(a))-\ln(1+\cos(b)) \\ &= J \end{aligned}$$

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