2012 Leaving Cert Higher Level
Official Sample Paper 1

Section A	**Concepts and Skills**	**150 marks**

Question 1 **(25 marks)**

(a) $w = -1 + \sqrt{3}i$ is a complex number, where $i^2 = -1$.

(i) Write w in polar form.

We have $|w| = \sqrt{(-1)^2 + \sqrt{3}^2} = \sqrt{4} = 2$. Also, if $\arg(w) = \theta$, then $\tan(\theta) = \frac{\sqrt{3}}{-1} = -\sqrt{3}$ and θ lies in the second quadrant (from the diagram). Therefore $\theta = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$radians. So

$$w = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

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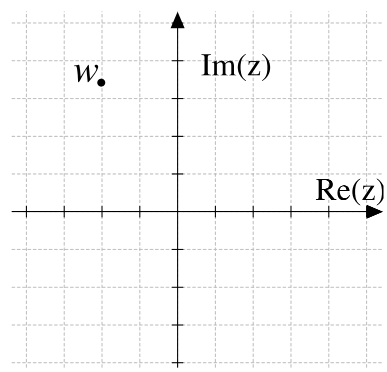


(ii) Use De Moivre's Theorem to solve the equation $z^2 = -1 + \sqrt{3}i$. Give your answer(s) in rectangular form.

Suppose that the polar form of z is given by $z = r(\cos\theta + i\sin\theta)$. Then

$$[r(\cos\theta + i\sin\theta)]^2 = 2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$$

By De Moivre's Theorem this is equivalent to

$$r^2(\cos(2\theta) + i\sin(2\theta)) = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right).$$

Therefore $r^2 = 2$ and $2\theta = \frac{2\pi}{3} + 2n\pi$, $n \in \mathbb{Z}$. So $r = \sqrt{2}$ and $\theta = \frac{\pi}{3} + n\pi$. We get two distinct solutions (corresponding to $n = 0$ and $n = 1$).

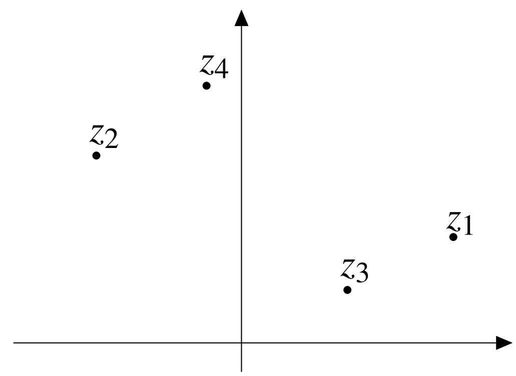
$$z_1 = \sqrt{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \sqrt{2}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{1}{\sqrt{2}} + i\sqrt{\frac{3}{2}}$$

and

$$z_2 = \sqrt{2}\left(\cos(\frac{\pi}{3} + \pi) + i\sin(\frac{\pi}{3} + \pi)\right) = \sqrt{2}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -\frac{1}{\sqrt{2}} - i\sqrt{\frac{3}{2}}$$

(b) Four complex numbers z_1, z_2, z_3 and z_4 are shown on the Argand diagram. They satisfy the following conditions:

$$z_2 = iz_1$$
$$z_3 = kz_1, \text{ where } k \in \mathbb{R}$$
$$z_4 = z_2 + z_3$$



The same scale is used on both axes.

(i) Identify which number is which by labelling the points on the diagram.

(ii) Write down the approximate value of k.

Answer: $\frac{1}{2}$

Explanation: Multiplication by i rotates a complex number by 90° anticlockwise about the origin - so z_2 is obtained by rotating z_1 through 90° about the origin.

Since $z_3 = kz_1$, we must have 0, z_1 and z_3 being collinear.

Since $z_4 = z_2 + z_3$, we must have 0, z_2, z_4 and z_3 forming a parallelogram.

Question 2 **(25 marks)**

(a) **(i)** Prove by induction that, for any n, the sum of the first n natural numbers is $\dfrac{n(n+1)}{2}$.

First we check that the statement is true for $n = 1$. The sum of the first 1 natural numbers is 1, and when $n = 1$ we have $\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$. So the statement is true for $n = 1$. Now suppose that the statement is true for some $n \geq 1$. So

$$1+2+\cdots+n = \frac{n(n+1)}{2}.$$

Now, add $n+1$ to both sides and we get

$$\begin{aligned}1+2+\cdots+n+(n+1) &= \frac{n(n+1)}{2}+(n+1)\\ &= \frac{n(n+1)}{2}+\frac{2(n+1)}{2}\\ &= \frac{n(n+1)+2(n+1)}{2}\\ &= \frac{(n+2)(n+1)}{2}\\ &= \frac{(n+1)(n+2)}{2}\end{aligned}$$

So the sum of the first $n+1$ natural numbers is $\frac{(n+1)((n+1)+1)}{2}$, which completes the induction step. Therefore, by induction, the statement is true for all natural numbers n.

(ii) Find the sum of all the natural numbers from 51 to 100, inclusive.

By part (i), we know that

$$1+2+\cdots+100 = \frac{100(101)}{2} = 5050.$$

We also know that

$$1+2+\cdots+50 = \frac{50(51)}{2} = 1275.$$

Subtracting the second equation from the first yields

$$51+52+\cdots+100 = 5050-1275 = 3775.$$

(b) Given that $p = \log_c x$, express $\log_c \sqrt{x} + \log_c(cx)$ in terms of p.

We know that

$$\log_c \sqrt{x} = \log_c x^{\frac{1}{2}} = \frac{1}{2}\log_c x = \frac{1}{2}p$$

using the power law for logarithms.
Also,

$$\log_c(cx) = \log_c c + \log_c x = \log_c c + p$$

using the product rule for logarithms.
But $\log_c c = 1$ since $c^1 = c$. Therefore

$$\log_c \sqrt{x} + \log_c(cx) = \frac{1}{2}p + 1 + p = \frac{3p}{2} + 1.$$

Question 3 **(25 marks)**

A cubic function f is defined for $x \in \mathbb{R}$ as

$$f : x \mapsto x^3 + (1-k^2)x + k, \text{ where } k \text{ is a constant.}$$

(a) Show that $-k$ is a root of f.

Substituting $-k$ for x we obtain

$$\begin{aligned} f(-k) &= (-k)^3 + (1-k^2)(-k) + k \\ &= -k^3 - k + k^3 + k \\ &= 0 \end{aligned}$$

Therefore $-k$ is a root of f.

(b) Find, in terms of k, the other two roots of f.

Since $-k$ is a root of f we know, by the Factor Theorem, that $(x+k)$ is a factor of $f(x)$. Now we carry out long division to find the other factor.

$$
\begin{array}{rcccccccc}
 & x^2 & - & kx & + & 1 & & \\
\cline{2-8}
x+k \,\big) & x^3 & + & 0x^2 & + & (1-k^2)x & + & k \\
 & x^3 & + & kx^2 & & & & \\
\cline{2-4}
 & & - & kx^2 & + & (1-k^2)x & & \\
 & & - & kx^2 & - & k^2x & & \\
\cline{3-6}
 & & & & & x & + & k \\
 & & & & & x & + & k \\
\cline{6-8}
 & & & & & & & 0
\end{array}
$$

So

$$x^3+(1-k^2)x+k=(x+k)(x^2-kx+1).$$

Therefore the other two roots of f are solutions of the equation

$$x^2-kx+1=0.$$

Using the quadratic formula we get

$$x=\frac{k\pm\sqrt{k^2-4}}{2}.$$

So the other two roots of f are

$$\frac{k+\sqrt{k^2-4}}{2} \text{ and } \frac{k-\sqrt{k^2-4}}{2}.$$

(c) Find the set of values of k for which f has exactly one real root.

From the solution to part (b), we see that f has exactly one real root if and only if $k^2-4<0$. This is equivalent to $k^2<4$ or

$$-2<k<2$$

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Question 4 **(25 marks)**

(a) Solve the simultaneous equations:

$$\begin{aligned} 2x+8y-3z &= -1 \\ 2x-3y+2z &= 2 \\ 2x+y+z &= 5. \end{aligned}$$

We can subtract the second equation from the first:

$$\begin{array}{rcl} 2x+8y-3z &=& -1 \\ 2x-3y+2z &=& 2 \\ \hline 11y-5z &=& -3 \end{array}$$

Similarly, we subtract the third equation from the first:

$$\begin{array}{rcl} 2x+8y-3z &=& -1 \\ 2x+y+z &=& 5 \\ \hline 7y-4z &=& -6 \end{array}$$

Now we solve the simultaneous equations

$$\begin{array}{rcl} 11y-5z &=& -3 \\ 7y-4z &=& -6 \end{array}$$

Multiply the first by 7, the second by 11 and subtract:

$$\begin{array}{rcl} 77y-35z &=& -21 \\ 77y-44z &=& -66 \\ \hline 9z &=& 45 \end{array}$$

Therefore $z = \frac{45}{9} = 5$. Now substitute $z = 5$ into $7y-4z = -6$ to get $7y-4(5) = -6$ or $7y = -6+20 = 14$ Therefore $y = 2$.
Finally substitute $y = 2$ and $z = 5$ into $2x+8y-3z = -1$ to get $2x+8(2)-3(5) = -1$ or $2x = -1-8(2)+3(5) = -2$ So $x = -1$.
So the solution is

$$x = -1, y = 2, z = 5.$$

Now we can check this by substituting into the original equations and verifying that they are all true:

$$\begin{array}{rcl} 2(-1)+8(2)-3(5) &=& -1 \\ 2(-1)-3(2)+2(5) &=& 2 \\ 2(-1)+(2)+(5) &=& 5. \end{array}$$

(b) The graphs of the functions $f : x \mapsto |x-3|$ and $g : x \mapsto 2$ are shown in the diagram.

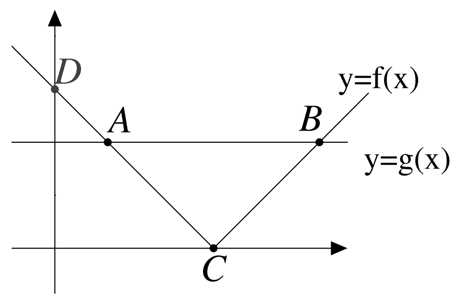
(i) Find the co-ordinates of the points A, B, C and D.

D is on the y-axis, so its x-co-ordinate is 0. Now $f(0) = |0-3| = |-3| = 3$. So $D = (0,3)$.
$C = (,0)$ (on the x-axis), so we solve $|x-3| = 0$ to find the x-co-ordinate. Now $|x-3| = 0 \Leftrightarrow x-3 = 0 \Leftrightarrow x = 3$. So $C = (3,0)$.
A and B both have y-co-ordinate 2, so we solve $|x-3| = 2$. Now $|x-3| = 2 \Leftrightarrow \pm(x-3) = 2$. So either

$$(x-3) = 2 \text{ or } -(x-3) = 2.$$

In the first case $x = 5$ and in the second case $-x+3 = 2$ or $x = 1$. So $A = (1,2)$ and $B = (5,2)$.

$$A = (1,2) \quad B = (5,2)$$
$$C = (3,0) \quad D = (0,3)$$



(ii) Hence, or otherwise, solve the inequality $|x-3| < 2$.

The solution set of the inequality corresponds to the values of x for which the graph of f is below the graph of g. From the diagram and calculations above, we see that the solution set is

$$1 < x < 5.$$

Question 5 **(25 marks)**

A is the closed interval $[0,5]$. That is, $A = \{x | 0 \le x \le 5, x \in \mathbb{R}\}$.
The function f is defined on A by:

$$f : A \to \mathbb{R} : x \mapsto x^3 - 5x^2 + 3x + 5.$$

(a) Find the absolute maximum and minimum values of f.

First we calculate the derivative of f. So

$$f'(x) = 3x^2 - 5(2x) + 3(1) + 0 = 3x^2 - 10x + 3$$

Now we solve $f'(x) = 0$ to find the stationary points of f. Now

$$3x^2 - 10x + 3 = 0 \Leftrightarrow x = \frac{10 \pm \sqrt{10^2 - 36}}{2(3)} \Leftrightarrow x = \frac{10 \pm 8}{6}.$$

So the stationary points of $f(x)$ are given by

$$x = 3 \text{ or } x = \frac{1}{3}.$$

Now both of these lie inside A. So the maximum/minimum value of $f(x)$ occurs at one $x = 0, \frac{1}{3}, 3$ or 5. Now

$$\begin{aligned} f(0) &= 5 \\ f(\tfrac{1}{3}) &= \frac{1}{27} - \frac{5}{9} + 1 + 5 = \frac{148}{27} \\ f(3) &= 27 - 5(9) + 3(3) + 5 = -4 \\ f(5) &= 125 - 5(25) + 3(5) + 5 = 20 \end{aligned}$$

Therefore

The maximum value of f is 20

and

The minimum value of f is -4

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(b) State whether f is injective. Give a reason for your answer.

f is NOT injective.
Reason: We see from our solution to part (a) that $f(\frac{1}{3}) > 0 > f(3)$ - in particular the sign of $f(x)$ changes between $x = \frac{1}{3}$ and $x = 3$. This means that there is some $a < 3$ such that $f(a) = 0$
Similarly, since $f(3) < 0 < f(5)$, there is some $b > 3$ such that $f(b) = 0$.
So $a \neq b$ since $a < 0$ and $b > 0$, but $f(a) = f(b) = 0$. Therefore f is not injective.

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Question 6 **(25 marks)**

(a) **(i)** Write down three distinct anti-derivatives of the function

$$g : x \mapsto x^3 - 3x^2 + 3, \quad x \in \mathbb{R}.$$

1. $G : x \mapsto \frac{1}{4}x^4 - x^3 + 3x.$
2. $G : x \mapsto \frac{1}{4}x^4 - x^3 + 3x + 1.$
3. $G : x \mapsto \frac{1}{4}x^4 - x^3 + 3x + 6.$

(ii) Explain what is meant by the indefinite integral of a function f.

The indefinite integral of f is the general form of a function whose derivative is f.

Alternative answer: The indefinite integral of f is $F(x) + C$ where $F' = f$ and C is constant (the constant of integration).

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(iii) Write down the indefinite integral of g, the function in part **(i)**.

Answer: $$\int g(x)dx = \frac{1}{4}x^4 - x^3 + 3x + C.$$

(b) **(i)** Let $h(x) = x \ln x$, for $x \in \mathbb{R}$, $x > 0$.
Find $h'(x)$.

Using the product rule we see that

$$h'(x) = (x)' \ln x + x(\ln x)'.$$

But $(x)' = 1$ and $(\ln x)' = \frac{1}{x}$. Therefore

$$\begin{aligned} h'(x) &= (1)\ln x + x\left(\frac{1}{x}\right) \\ &= \ln x + 1. \end{aligned}$$

(ii) Hence, find $\int \ln x dx$.

We know that $h'(x) = \ln x + 1$. Also, we know that $(x)' = 1$. So if $F(x) = h(x) - x$, then

$$F'(x) = h'(x) - (x)' = \ln x + 1 - 1 = \ln x.$$

Therefore $\int \ln x\, dx = F(x) + c$. But $F(x) = h(x) - x = x\ln x - x$. Therefore

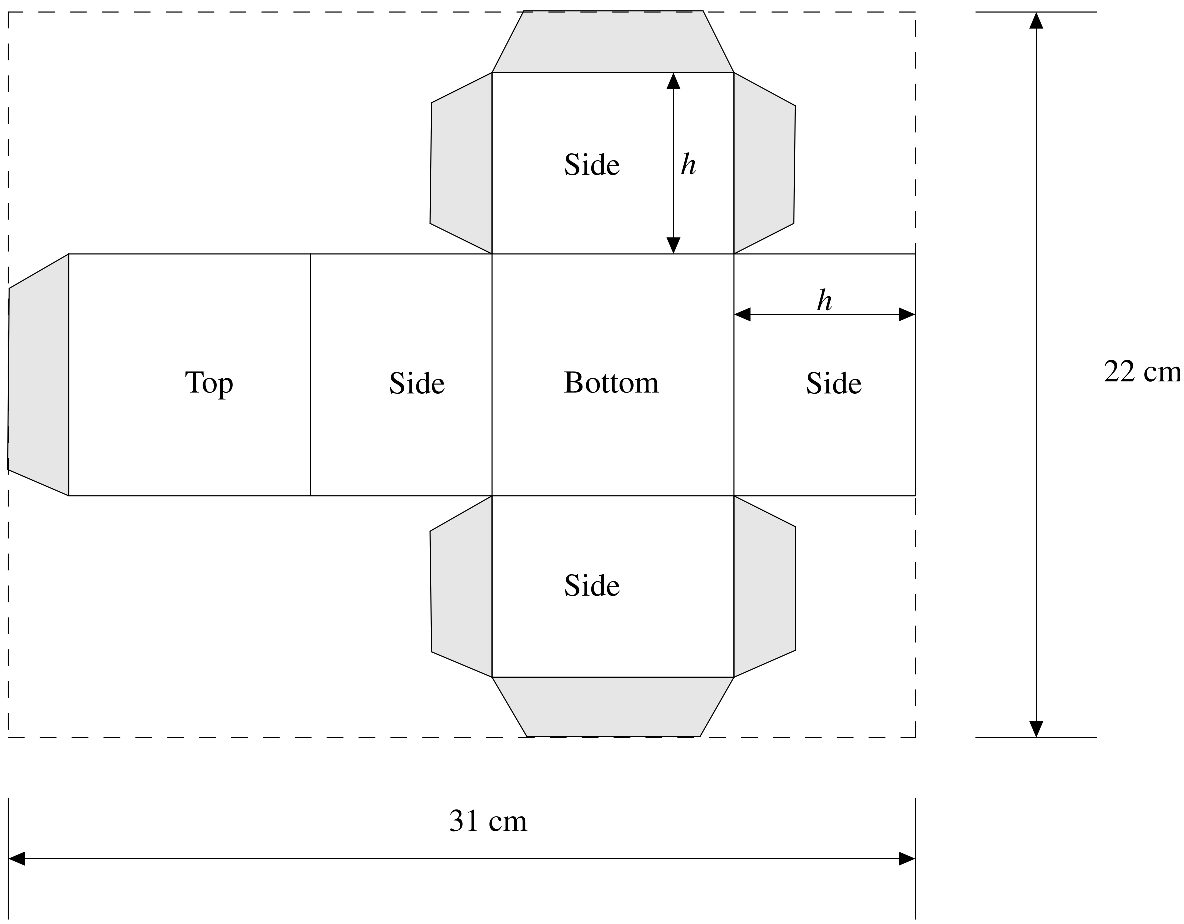
$$\int \ln x\, dx = x\ln x - x + C.$$

Section B — Contexts and Applications — 150 marks

Question 7 **(50 marks)**

A company has to design a rectangular box for a new range of jellybeans. The box is to be assembled from a single piece of cardboard, cut from a rectangular sheet measuring 31 cm by 22 cm. The box is to have a capacity (volume) of 500 cm^3 .

The net for the box is shown below. The company is going to use the full length and width of the rectangular piece of cardboard. The shaded areas are flaps of width 1 cm which are needed for assembly. The height of the box is h cm, as shown on the diagram.



(a) Write the dimensions of the box, in centimetres, in terms of *h*.

Let l be the length of the box and let w be the width of the box, both in centimetres. Then by adding up dimensions as we move left to right across the diagram above, we see that $1+l+h+l+h=31$. Therefore, by isolating l in this equation we obtain

$$l = 15 - h.$$

Going top to bottom, we see that $1+h+w+h+1=22$ and by isolating w, we see that

$$w = 20 - 2h.$$

Therefore

height =	h cm
length =	$15-h$ cm
width =	$20-2h$ cm

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(b) Write an expression for the capacity of the box in cubic centimetres, in terms of h.

Capacity = length $\times$ width $\times$ height $= (15-h)(20-2h)h = 2h^3 - 50h^2 + 300h\text{cm}^3$.

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(c) Show that the value of h that gives a box with a square bottom will give the correct capacity.

The bottom of the box is square if and only if length = width. In other words, if and only if $15-h=20-2h$. This is equivalent to $h=5$. From the solution to part (b), we calculate that, when $h=5$, the capacity of the box will be $(15-5)(20-2(5))5 = 10(10)(5) = 500\text{cm}^3$, as required.

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(d) Find, correct to one decimal place, the other value of h that gives a box of the correct capacity.

We must solve $2h^3 - 50h^2 + 300h = 500$, or

$$2h^3 - 50h^2 + 300h - 500 = 0.$$

From part (c), we know that $h = 5$ is one solution. Therefore, by the Factor Theorem, $(h-5)$ is a factor of $2h^3 - 50h^2 + 300h - 500$. Factorising yields

$$2h^3 - 50h^2 + 300h - 500 = (h-5)(2h^2 - 40h + 100).$$

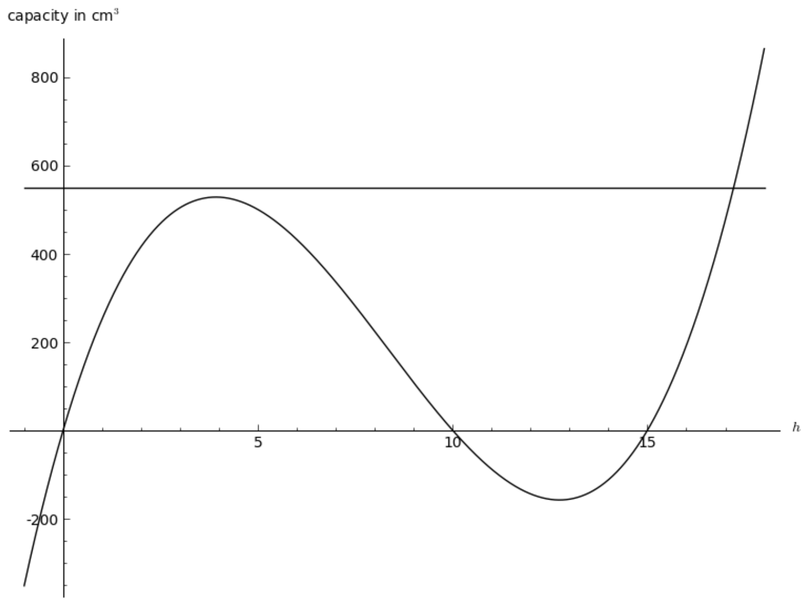
Now, we solve $2h^2 - 40h + 100 = 0$ using the quadratic formula. So

$$h = \frac{40 \pm \sqrt{40^2 - 4(2)(100)}}{2(2)} = \frac{40 \pm \sqrt{800}}{4} = 10 \pm \sqrt{50}$$

So, correct to one decimal place, $h = 17.1$ or $h = 2.9$.
Now, however, we observe that since the length of the box is $15 - h$, we must have $15 - h > 0$ or $h < 15$. Therefore $h \neq 17.1$. So the other value of h that gives the correct capacity is 2.9cm.

(e) The client is planning a special "10% extra free" promotion and needs to increase the capacity of the box by 10%. The company is checking whether they can make this new box from a piece of cardboard the same size as the original one (31cm × 22cm). They draw the graph below to represent the box's capacity as a function of h. Use the graph to explain why it is not possible to make the larger box from such a piece of cardboard.

Explanation:



The capacity of the new box will be $1.1 \times 500 = 550\text{cm}^3$. On the diagram above we have drawn a horizontal line representing the equation

$$\text{Capacity} = 550.$$

We can see from the diagram that this horizontal line only meets the cubic curve at one point and that the h-co-ordinate of that point is greater than 15.
However, as we observed above, for any box constructed as described in the question, we must have $h < 15$. Therefore it is not possible to make the bigger box from the same piece of cardboard as before.

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Question 8 **(50 marks)**

Pádraig is 25 years old and is planning for his pension. He intends to retire in forty years' time, when he is 65. First, he calculates how much he wants to have in his pension fund when he retires. Then, he calculates how much he needs to invest in order to achieve this. He assumes that, in the long run, money can be invested at an inflation-adjusted annual rate of 3%. Your answers throughout this question should therefore be based on a 3% annual growth rate.

(a) Write down the present value of a future payment of €20,000 in one year's time.

We have

$$P = \frac{F}{1+i} = \frac{20000}{1.03} = 19417.48$$

So the present value is €19417.48 to the nearest cent.

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(b) Write down, in terms of t, the present value of a future payment of €$20,000$ in t years' time.

We have

$$P = \frac{F}{(1+i)^t} = \frac{20000}{(1.03)^t}.$$

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(c) Pádraig wants to have a fund that could, from the date of his retirement, give him a payment of €$20,000$ at the start of each year for 25 years. Show how to use the sum of a geometric series to calculate the value, on the date of retirement, of the final fund required.

Using the solution to part (b), we see that the amount required on the date of requirement is given by

$$A \quad = \quad 20000 + \frac{20000}{1.03} + \cdots \frac{20000}{(1.03)^{24}}$$

Using the notation of the formula on page 22 of the Formula and Tables book, we have a geometric series with $a = 20000$, $r = \frac{1}{1.03}$ and $n = 25$. Therefore

$$A = \frac{20000\left(1 - \left(\frac{1}{1.03}\right)^{25}\right)}{1 - \frac{1}{1.03}}$$

Using a calculator we obtain

$$A = €358711$$

to the nearest euro.

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(d) Pádraig plans to invest a fixed amount of money every month in order to generate the fund calculated in part **(c)**. His retirement is $40 \times 12 = 480$ months away.

(i) Find, correct to four significant places, the rate of interest per month that would, if paid and compounded annually, be equivalent to an effective annual rate of 3%.

We must solve $(1+i)^{12} = 1.03$. So $(1+i) = \sqrt[12]{1.03}$. Therefore

$$i = \sqrt[12]{1.03} - 1 = 0.002466$$

correct to 4 significant places. So the answer is

$$0.2466\%.$$

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(ii) Write dowm, in terms of n and P, the value on the retirement date of a payment of €P made n months before the retirement date.

Using the formula on page 30 of the Formula and Tables booklet we obtain

$$P(1.002466)^n.$$

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(iii) If Pádraig makes 480 equal payments of €P from now until his retirement, what value of P will give him the fund he requires?

We must solve

$$P(1.002466)^{480}+P(1.002466)^{479}+\cdots+P(1.002466)=358711$$

or

$$P(1.002466+(1.002466)^2+\cdots+(1.002466)^{480})=358711$$

Using the formula for the sum of a geometric series, we obtain

$$P\left(\frac{1.002466(1-(1.002466)^{480})}{1-1.002466}\right)=358711$$

or

$$P(919.38)=358711.$$

Therefore $P=\frac{358711}{919.38}=€390.17$ to the nearest cent.

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(e) If Pádraig waits for ten years before starting his pension fund, how much will he then have to pay each month in order to generate the same pension fund?

Now the number of months until his retirement date is $30\times 12=360$. So as above we must solve

$$P(1.002466)^{360}+P(1.002466)^{359}+\cdots+P(1.002466)=358711$$

or

$$P(1.002466+(1.002466)^2+\cdots+(1.002466)^{360})=358711$$

Using the formula for the sum of a geometric series, we obtain

$$P\left(\frac{1.002466(1-(1.002466)^{360})}{1-1.002466}\right)=358711$$

or

$$P(580.11)=358711.$$

Therefore, in this case, $P=\frac{358711}{580.11}=€618.35$ to the nearest cent.

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Question 9 **(50 marks)**

(a) Let $f(x)=-0.5x^2+5x-0.98$ where $x\in\mathbb{R}$.

(i) Find the value of $f(0.2)$

Substituting 0.2 for x gives

$$f(0.2) = -0.5(0.2)^2 + 5(0.2) - 0.98 = -0.5(0.04) + 1 - 0.98 = 0$$

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(ii) Show that f has a local maximum point at $(5, 11.52)$.

First we calculate the derivative of f:

$$f'(x) = -0.5(2x) + 5(1) - 0 = -x + 5.$$

Now $f'(5) = -5 + 5 = 0$. Therefore $x = 5$ is a stationary point.
Now

$$f''(x) = -1.$$

So $f''(5) = -1 < 0$. That means that $x = -5$ is a local maximum. Finally,

$$f(5) = -0.5(5^2) + 5(5) - 0.98 = 11.52.$$

Therefore the graph of f has a local maximum point at $(5, 11.52)$.

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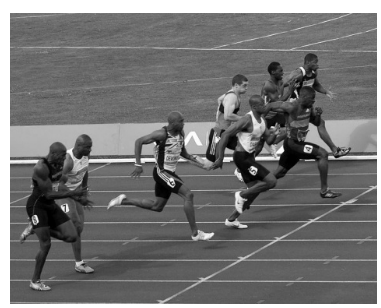
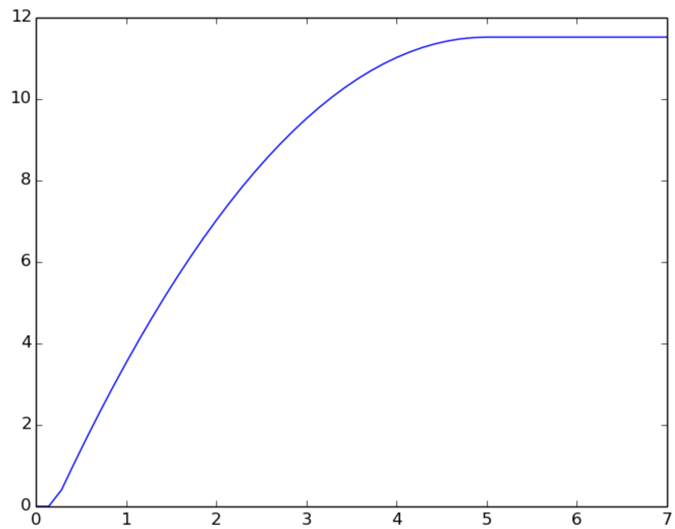
(b) A sprinter's velocity over the course of a particular 100 metre race is approximated by the following model, where v is the velocity in metres per second, and t is the time in seconds from the starting signal:

$$v(t) = \begin{cases} 0, & \text{for } 0 \leq t \leq 0.2 \\ -0.5t^2 + 5t - 0.98, & \text{for } 0.2 \leq t \leq 5 \\ 11.52, & \text{for t } \geq 5 \end{cases}$$

Note that the function in part **(a)** is relevant to $v(t)$ above.



(i) Sketch the graph of v as a function of t for the first 7 seconds of the race.



Note that between $t = 0$ and $t = 0.2$ the graph is just a horizontal line along the t-axis. Likewise, for $t \geq 5$ the graph is a horizontal line at height $v = 11.52$. In between $t = 0.2$ and $t = 5$ the function is a quadratic so the graph must be a parabola. We can sketch this by evaluating the function at three or four points. For example $v(1) = 3.52$, $v(2) = 7.02$, $v(3) = 9.52$ and $v(4) = 11.02$. So we plot the points $(1, 3.52)$, $(2, 7.02)$, $(3, 9.52)$ and $(4, 11.02)$ and then join them by a smooth curve. Make sure that this parabolic arc starts at $(0.2, 0)$ and ends at $(5, 11.52)$.

(ii) Find the distance travelled by the sprinter in the first 5 seconds of the race.

The distance travelled in the first 5 seconds of the race is given by

$$\int_0^5 v(t) \quad dt.$$

Now

$$\begin{aligned}
\int_0^5 v(t) \quad dt &= \int_0^{0.2} v(t) \quad dt + \int_{0.2}^5 v(t) \quad dt \\
&= \int_0^{0.2} 0 \quad dt + \int_{0.2}^5 (-0.5t^2 + 5t - 0.98) \quad dt \\
&= 0 + \int_{0.2}^5 (-0.5t^2 + 5t - 0.98) \quad dt \\
&= \int_{0.2}^5 (-0.5t^2 + 5t - 0.98) \quad dt \\
&= \frac{-0.5t^3}{3} + \frac{5t^2}{2} - 0.98t \Big|_{0.2}^5 \\
&= \frac{0.5(5^3)}{3} + \frac{5(5^2)}{2} - 0.98(5) - \left(\frac{0.5(0.2^3)}{3} + \frac{5(0.2^2)}{2} - 0.98(0.2)\right) \\
&= 36.864
\end{aligned}$$

So the sprinter travels 36.864 metres in the first 5 seconds of the race.

(iii) Find the sprinter's finishing time for the race. Give your answer correct to two decimal places.

We have just seen that the sprinter travels 36.864 metres in the first 5 seconds of the race. So he has 63.136metres left to travel to complete the race at that point. Also after 5 seconds, his velocity is a constant 11.52 metres per second. Therefore it will take him a further $\frac{63.136}{11.52}$ seconds to complete the race. Now $\frac{63.136}{11.52} = 5.48$ correct to two decimal places. So his total time is $5 + 5.48 = 10.48$ seconds, correct to two decimal places.

After 7 seconds the sprinter's velocity doesn't change. So the first step is to calculate how far he has travelled in the first 7 seconds of the race. As in part (ii) this is equal to the value of the defnite integral

$$\int_{0.2}^{7} (-0.5t^2 + 5t - 0.98) \quad dt.$$

Now

$$\begin{aligned}
\int_{0}^{7} (-0.5t^2 + 5t - 0.98) \quad dt &= \frac{-0.5t^3}{3} + \frac{5t^2}{2} - 0.98t \Big|_{0.2}^{7} \\
&= \frac{0.5(7^3)}{3} + \frac{5(7^2)}{2} - 0.98(7) \\
&\quad - \left(\frac{0.5(0.2^3)}{3} + \frac{5(0.2^2)}{2} - 0.98(0.2) \right) \\
&= 58.571
\end{aligned}$$

So he travels 58.571 metres in 7 seconds. Therefore, he has $100 - 58.571 - 41.429$ metres left to travel at that point. His velocity for rest of the race is 11.52 metres per second. Therefore it will take him another $\frac{41.429}{11.52} = 3.596$ seconds to complete the race. So his total time for the race is $7 + 3.596 = 10.596$. So it takes him 10.60 seconds to finish the race, correct to two decimal places.

(c) A spherical snowball is melting at a rate proportional to its surface area. That is, the rate at which its volume is decreasing at any instant is proportional to its surface area at that instant.

(i) Prove that the radius of the snowball is decreasing at a constant rate.

Let t be time. Let r be the radius, A the surface area and V the volume of the snowball. From the Formula and Tables booklet we know that $A = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$. In particular,

$$\frac{dV}{dr} = \frac{4}{3}\pi\left(3r^2\right) = 4\pi r^2 = A.$$

Now we are told that the rate of change of volume with respect to time is proportional to the surface area. In other words,

$$\frac{dV}{dt} = kA \tag{1}$$

for some constant k. Clearly $k < 0$ since the volume of the snowball is decreasing as it melts. On the other hand, using the chain rule, we see that

$$\begin{aligned} \frac{dV}{dt} &= \frac{dV}{dr}\frac{dr}{dt} \\ &= A\frac{dr}{dt} \end{aligned} \tag{2}$$

Therefore by combining (1) and (2) , we see that

$$A\frac{dr}{dt} = kA.$$

Now dividing across by A yields

$$\frac{dr}{dt} = k$$

where k is a constant, as required.

(ii) If the snowball loses half of its volume in an hour, how long more will it take for it to melt completely?

Give your answer to the nearest minute.

Let r_0 be the initial radius and let r_2 be the radius after 1 hour.
So the initial volume is $\frac{4}{3}\pi r_0^3$. Therefore after one hour, the volume is $\frac{2}{3}\pi r_0^3$. Therefore

$$\frac{4}{3}\pi r_1^3 = \frac{2}{3}\pi r_0^2.$$

Therefore

$$\left(\frac{r_1}{r_0}\right)^3 = \frac{1}{2}$$

or

$$r_1 = \frac{1}{\sqrt[3]{2}} r_0.$$

Now the radius is decreasing at a constant rate and we have found that it takes 1 hour for it to decrease from r_0 to $\frac{1}{\sqrt[3]{2}} r_0$. Therefore the rate of change of the radius is $r_0 - \frac{1}{\sqrt[3]{2}} r_0$ units per hour.
Now the snowball will have melted completely when the radius reaches 0. So we calculate the time required to to change from r_0 to 0. This will be

$$\frac{\text{total change}}{\text{rate of change}} = \frac{r_0 - 0}{r_0 - \frac{1}{\sqrt[3]{2}} r_0} = \frac{1}{1 - \frac{1}{\sqrt[3]{2}}} \text{ hours.}$$

This is equal to 4.8473 hours (correct to four decimal places). So it will take 3.8473 more hours (after the 1 already elapsed) for it to melt completely.
Now 3.8473 hours is equal $3.8473 \times 60 = 230.84$.
So, to the nearest minute, it will take a further 231 minutes for the snowball to melt completely.